# Self-oscillations in ring Toda chains with negative friction

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We study here the different modes of self-oscillations in ring Toda chains with Rayleigh-type negative friction. Assuming that at small friction the shape of self-oscillations is close to one of the known Toda solitonlike solutions we use analytical methods in combination with numerical ones for study of the self-oscillations. We calculate explicitly for a Toda chain consisting of $N$ elements the $N+1$ different modes of self-oscillations. Among them two modes correspond to left and right rotations of the chain as a whole with a constant velocity Each of the other $N-1$ modes represents a combination of solitonlike oscillations and a rotation with a velocity depending on the mode number. Only for the mode corresponding to antiphase oscillations of the chain neighboring elements (such oscillations are possible for an even $N$) the constant component of the velocity is equal to zero.

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## I. INTRODUCTION

Chains of homogeneous oscillatory elements with exponential interaction between them were first studied by Toda [1–3]. Later it was found that Toda's equations are generative for many physical problems, for example, for the problem of self-synchronization of modes in lasers [4].

Since Toda chains are completely integrable systems [5], stationary ''waves'' akin to solitons in continuous media are possible in these chains. A partial solution of Toda's equations that describes such ''solitons'' were found in Ref. [2]. In practice, however, numerical tests show instability of these solutions with respect to small perturbations, in particular, to numerical or real noise. This is associated with the fact that even a very small deviation of a Hamiltonian system from integrable one can cause its stochastic behavior [6].

Waves similar to solitons are also possible in slightly dissipative systems with energy sources, in particular, in self-oscillatory ones. Solitonlike waves in self-oscillatory systems received the name autosolitons [20,7] or dissipative solitons [21,8]. In the last few years extensive literature on autosolitons appeared, including theoretical [7,8] and experimental works [9]. It is interesting that the presence of dissipation may stabilize solitonlike waves.

Autosolitons in dissipative ring Toda chains were considered in Refs. [10,11]. Different types of dissipative functions were studied [10,12]. One of the main results of these investigations lies in the fact that dissipation acts in a very specific way on the stability of solitonlike waves. Sometimes dissipation destroys these waves, sometimes it stabilizes them [8]. Electrical circuit implementations of Toda-like soliton systems were studied by Singer and Oppenheim [16] and by Makarov, del Rio *et al.* [17].

The aim of this paper is to calculate explicitly the different modes of self-oscillations in ring Toda chains using analytical methods in combination with numerical ones. We show that the chains consisting from $N$ elements possess $N+1$ different modes of self-oscillations which may be represented by elliptic functions. Among them are two modes which correspond to left and right rotations with a constant velocity and $N-1$ oscillatory modes.

## II. MODELS OF SELF-OSCILLATORY TODA CHAINS

Let us consider a chain consisting of $N$ balls of mass $m$ connected by nonlinear springs and closed in a ring. The distance between the neighboring balls

$$z_j = x_j - x_{j-1} \tag{1}$$

determines the strain of the $j$th spring (see Ref. [13]). In moderately general form equations of the chain considered having regard to small dissipative terms can be written as

$$m\ddot{x}_j - f(z_j) + f(z_{j+1}) = m\mu \sum_{i=1}^{N} g_{ji}(\mathbf{x},\dot{\mathbf{x}})\dot{x}_i \quad (j=1,2\ldots,N), \tag{2}$$

where

$$f(z) = -\alpha(1-e^{-\gamma z}) \tag{3}$$

is the nonlinear function describing the elasticity of the springs, $\mathbf{x}$ and $\dot{\mathbf{x}}$ denote the sets of the variables $x_k$ and $\dot{x}_k$, respectively, $g_{ji}(\mathbf{x},\dot{\mathbf{x}})$ are nonlinear functions involving positive constant constituents $a_{ji}$. Without the loss of generality the factor $\gamma$ may be put equal to unity. The functions $g_{ji}(\mathbf{x},\dot{\mathbf{x}})$ play the role of nonlinear friction factors. The terms $\mu a_{ji}\dot{x}_i$ describe linear negative friction resulting in the excitation of self-oscillations. For sufficiently small oscillation amplitudes each of $g_{ji}$ can be expanded in a Taylor series. Taking into account that terms of the expansion with even powers are of prime importance for the amplitude limitation, we may retain in the expansion only quadratic terms.

Subtracting pairwise Eqs. (2) we can rewrite them in the strain coordinates $z_j$ as

 

$$m\ddot{z}_j+f(z_{j-1})-2f(z_j)+f(z_{j+1})=m\mu\sum_{i=1}^{N}h_{ji}(\mathbf{z},\dot{\mathbf{z}})\dot{z}_i \quad (j=1,2,\ldots,N). \tag{4}$$

In the simplest case the functions $g_{ji}(\mathbf{x},\dot{\mathbf{x}})$ and $h_{ji}(\mathbf{z},\dot{\mathbf{z}})$ depend only on the index $j$, i.e., $\Sigma_{i=1}^{N}g_{ji}(\mathbf{x},\dot{\mathbf{x}})\dot{x}_i=g_j(x_j,\dot{x}_j)\dot{x}_j$ and $\Sigma_{i=1}^{N}h_{ji}(\mathbf{z},\dot{\mathbf{z}})\dot{z}_i=h_j(z_j,\dot{z}_j)\dot{z}_j$. We will assume that in the vicinity of the boundary of self-excitation where the amplitudes are small, the functions $g_j$ and $h_j$ may be approximated by quadratic forms similar to that used in the classical Rayleigh equation [13], namely,

$$g_j=a-\dot{x}_j^2, \tag{5}$$

$$h_j=b-\dot{z}_j^2. \tag{6}$$

Since the chain is closed in a ring, the conditions

$$x_{j+N}=x_j, \quad \dot{x}_{j+N}=\dot{x}_j \tag{7}$$

must be fulfilled for any $j$.

## III. SOLITONLIKE OSCILLATIONS OF THE CONSERVATIVE TODA CHAIN

Solitonlike oscillations in the chain described by Eq. (2) for $\mu=0$ were found by Toda [2]. Toda used the change of variables

$$\dot{y}_j=f(z_j). \tag{8}$$

With this change of variables Eqs. (2) for $\mu=0$ become

$$m\dot{x}_j-y_j+y_{j+1}=mC, \tag{9}$$

where $C$ is an arbitrary constant. Eliminating from Eqs. (1), (8), and (9), in view of Eq. (3), the variables $x_j$ and $z_j$, we find the following equations for $y_j$:

$$\ddot{y}_j=\frac{\alpha+\dot{y}_j}{m}(y_{j-1}-2y_j+y_{j+1}). \tag{10}$$

A partial solution of Eq. (10) can be sought in the form of a "running wave"

$$y_j(t)=\varphi(\xi_j), \tag{11}$$

where $\xi_j=\omega t-\beta j$, $\varphi$ is a periodic function with period $2\pi$, and $\beta$ is the phase shift between oscillations of neighboring elements. Substituting Eq. (11) into Eq. (10) we obtain the following equation for the function $\varphi(\xi_j)$:

$$m\omega^2\varphi''=(\alpha+\omega\varphi')[\varphi(\xi_j-\beta)-2\varphi(\xi_j)+\varphi(\xi_j+\beta)], \tag{12}$$

where the prime indicates differentiation with respect to $\xi_j$.

Toda showed that a solution of Eq. (12) can be expressed in terms of the Jacobi elliptic zeta function [18] as

$$\varphi(\xi_j)=A\,\mathrm{zn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right), \tag{13}$$

where

$$\mathrm{zn}(\vartheta,k)=\int_0^{\vartheta}[\mathrm{dn}(x,k)]^2dx-\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\vartheta$$

is the Jacobi elliptic zeta function, and $\mathbf{K}(k)$ and $\mathbf{E}(k)$ are the full elliptic integrals of the first kind and of the second kind, respectively. Substituting (13) into Eq. (12) we find the following equations relating the amplitude $A$, the modulus of the elliptic function $k$, the frequency $\omega$ and the phase shift $\beta$:

$$A=\frac{m\omega\mathbf{K}(k)}{\pi}, \tag{14}$$

$$\omega=\frac{\pi\omega_0}{2\mathbf{K}(k)}\left[1-\left(1-\frac{\mathbf{E}(k)}{\mathbf{K}(k)}\right)\mathrm{sn}^2\left(\frac{\mathbf{K}(k)}{\pi}\beta,k\right)\right]^{-1/2}\times\mathrm{sn}\left(\frac{\mathbf{K}(k)}{\pi}\beta,k\right), \tag{15}$$

where $\omega_0=2\sqrt{\alpha/m}$. It is easy to verify that Eq. (15) for small $k$ reduces to the dispersion equation for the corresponding linear chain.

It follows from Eqs. (9) and (13) that the velocity of the $j$th ball is

$$\dot{x}_j(\xi_j)=F(\xi_j,k,\omega)+C, \tag{16}$$

where

$$F(\xi_j,k,\omega)=\frac{A}{m}\left[\mathrm{zn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right)-\mathrm{zn}\left(\frac{\mathbf{K}(k)}{\pi}(\xi_j-\beta),k\right)\right]. \tag{17}$$

It is easily seen that $\dot{x}_j(t)$ is a periodic function of $\xi_j$ with period $2\pi$.

The strain of the $j$th spring can be found from Eq. (11) and the expression (13). As a result we obtain the following equation:

$$\alpha(1-e^{-z_j})=\frac{A\omega}{\pi}\left[\mathbf{E}(k)-\mathbf{K}(k)\mathrm{dn}^2\left(\frac{\mathbf{K}(k)}{\pi}\xi,k\right)\right]. \tag{18}$$

Taking account of Eq. (14), we find from Eq. (18):

$$z_j(\xi_j)=-\ln\left\{1-\frac{4\omega^2\mathbf{K}^2(k)}{\omega_0^2\pi^2}\left[\frac{\mathbf{E}(k)}{\mathbf{K}(k)}-\mathrm{dn}^2\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right)\right]\right\}. \tag{19}$$

It is evident that $z_j$ is also a periodic function of $\xi_j$ with period $2\pi$.

We note that $\mathbf{E}(k)\to 1$, $\mathbf{K}(k)\to\ln(4/\sqrt{1-k^2})$, $\mathrm{dn}\,\vartheta\to 1/\cosh\vartheta$, and $\mathrm{zn}\,\vartheta\to\tanh\vartheta-\vartheta/\mathbf{K}(k)$ as $k\to 1$. From this it follows that, for $k\to 1$,

$$\dot{x}_j(\xi_j) \approx \frac{\omega_0 \sqrt{\mathbf{K}(k)}}{2} \left[ \sum_{n=-\infty}^{\infty} \tanh\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi) \right) - \tanh\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi - \beta) \right) - \frac{\beta}{\pi} \right] + C. \quad (20)$$

The formula (20) allows us to calculate analytically $x_j(\xi_j)$ for $k$ close to 1. Integrating Eq. (20) over $t$ we find

$$x_j(\xi_j) \approx \sum_{n=-\infty}^{\infty} \ln\left[ \cosh\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi) \right) \times \cosh^{-1}\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi - \beta) \right) \right] - \frac{\beta \mathbf{K}(k)}{\pi^2} \xi_j - x_0 + Ct, \quad (21)$$

where

$$x_0 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} \ln\left[ \cosh\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi) \right) \times \cosh^{-1}\left( \frac{\mathbf{K}(k)}{\pi}(\xi_j + 2n\pi - \beta) \right) \right] - \frac{\beta \mathbf{K}(k)}{\pi^2} \xi_j d\xi_j .$$

From the conditions (7) we find possible values of $\beta$:

$$\beta = \beta_n = \frac{2\pi n}{N}, \quad n = 0, \ldots, N. \quad (22)$$

Thus, the conservative chain from $N$ elements possesses $N+1$ different modes of oscillations. These modes differ from one another in shape, amplitude, frequency and phase shift between the oscillations of neighboring balls. It should be noted that the values $n=0$ and $n=N$ correspond to the solution $\dot{x}_j = C$ which responds to uniform rotation of the chain as a whole; the values $n=1$ and $n=N-1$ correspond to a single traveling soliton on the background of uniform rotation, the values $n=2$ and $n=N-2$ correspond to two traveling solitons, and so on. In the active chain described by Eqs. (2) the found modes each may generate the corresponding attractor. Examples of the dependencies of $\dot{x}_j$, $x_j$, and $z_j$ on $\xi = \omega t - \beta j$ are given in Fig. 1 for $k = 1 - 10^{-7}$ and two values of $\beta$. It can be seen that, as $\beta$ increases, ''light'' solitons are interchanged by ''dark'' ones.

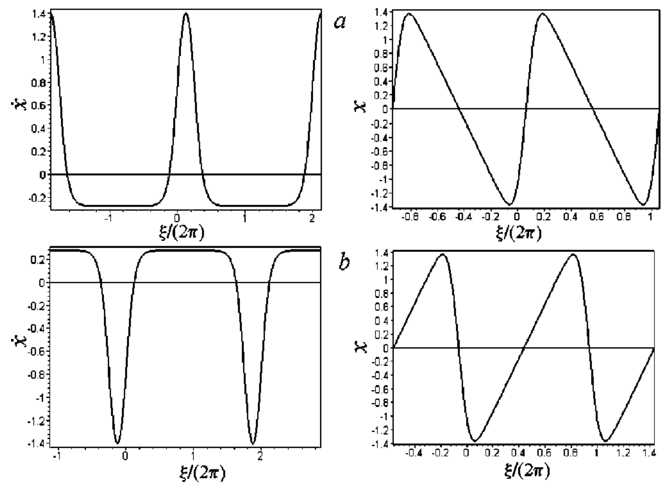


FIG. 1. Examples of the dependencies of $\dot{x}_j$, $x_j$, and $z_j$ on $\xi_j = \omega t - \beta j$ for $k = 1 - 10^{-7}$, (a) $\beta = \pi/4$ (''light'' solitons) and (b) $\beta = 7\pi/4$ (''dark'' solitons).

## IV. APPROXIMATE CALCULATION OF SELF-OSCILLATIONS

First we consider Eqs. (2) and set dissipative forces in the form (5) (such dissipative forces are equivalent to considered in Refs. [11,17]). In the case of small dissipation ($\mu \ll 1$) the solution (16) can be regarded as a generative one. This solution involves two arbitrary constants $k$ and $C$. $k$ is the modulus of the elliptic functions, which determines the amplitude and the shape of self-oscillations for the corresponding mode, and $C$ is the constant constituent of the ball velocity. To calculate $k$ and $C$, we require that the energy and momentum conservation laws should be fulfilled in the average for the oscillation period. To suit the first requirement, we can, for each $j$, multiply the $j$th equation from Eq. (2) by $\dot{x}_j$, then add all equations and average over time. As a result we obtain

$$\frac{d}{dt} \sum_{j=1}^{N} \int_0^{2\pi} \left( \frac{\dot{x}_j^2}{2} + \frac{1}{m} f(z_j) \right) d\xi_j = \mu \sum_{j=1}^{N} \int_0^{2\pi} (a - \dot{x}_j^2) \dot{x}_j^2 d\xi_j . \quad (23)$$

For calculating the integrals we should substitute Eq. (16), in view of Eq. (17), and

$$f(z_j) = \frac{A\omega}{\pi} \left[ \mathbf{K}(k) \mathrm{dn}^2\left( \frac{\mathbf{K}(k)}{\pi} \xi_j, k \right) - \mathbf{E}(k) \right]$$

into Eq. (23) and assume $k$ and $C$ to be constant. It is easily shown that all summands in Eq. (23) are identical. Eq. (23) is one of the truncated equations for $k$ and $C$.

Another truncated equation for $k$ and $C$ we find from the averaged momentum conservation law. Adding Eqs. (2), taking into account that, as follows from the momentum conservation law for the conservative chain $\Sigma_{j=1}^{N}[f(z_{j+1}) - f(z_j)] = 0$ and $\Sigma_{j=1}^{N} \partial F(\xi_j, k)/\partial \xi_j = 0$, and averaging over time we obtain

$$\sum_{j=1}^{N} \left( \frac{\partial \bar{F}}{\partial k} \frac{dk}{dt} + \frac{dC}{dt} \right) = \frac{\mu}{2\pi} \sum_{j=1}^{N} \int_0^{2\pi} (a - \dot{x}_j^2) \dot{x}_j d\xi_j , \quad (24)$$

where

$$\bar{F} = \frac{1}{2\pi} \int_0^{2\pi} F(\xi_j, k, \omega) d\xi_j . \quad (25)$$

Similar to Eq. (23), in Eq. (24) all summands are also identical.

Since the calculation of the left-hand sides of Eqs. (23) and (24) are rather complicated, we restrict ourselves to calculations of only steady-state values of $k$ and $C$ for the different modes of oscillations. Equations for these values can be found by equating the right-hand sides of Eqs. (23) and (24) to zero, i.e.,

$$\int_0^{2\pi}(a-\dot{x}_j^2)\dot{x}_j^2 d\xi_j=0, \tag{26}$$

$$\int_0^{2\pi}(a-\dot{x}_j^2)\dot{x}_j d\xi_j=0. \tag{27}$$

Substituting Eqs. (16),(17) into Eq. (27) we obtain a cubic equation for $C$ which can be written as

$$C^3+3pC+2q=0, \tag{28}$$

where $p=r_2-a/3,\ q=r_3/2$, and

$$r_n=\frac{A^n}{2\pi m^n}\int_0^{2\pi}\left[\mathrm{zn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right)-\mathrm{zn}\left(\frac{\mathbf{K}(k)}{\pi}(\xi_j-\beta),k\right)\right]^n d\xi_j. \tag{29}$$

According to Cardano's formula a real root of Eq. (28) is

$$C=(\sqrt{q^2+p^3}-q)^{1/3}-(\sqrt{q^2+p^3}+q)^{1/3}. \tag{30}$$

Taking account of Eqs. (29) and (28) we can rewrite Eq. (26) as

$$ar_2-r_4-3r_3C-3r_2C^2=0. \tag{31}$$

By substituting Eqs. (29) and (30) into Eq. (31) we find an equation for $k$ which can be solved by means of graphical displays. The results of the calculations for $N=8$ are illustrated in Fig. 2, where the dependencies of the steady-state values of $k$, of the constant constituent of velocity $C$, and of the frequencies $\omega$ are shown for two values of $a$ and $\omega_0$. We see that the values of $k$ and $|C|$ decrease monotonically as $n$ increases from 1 to 4 (or decreases from 7 to 4). The values of $C$ and $k$ found nearly coincide with those calculated from the results of direct computation of the initial equations (2) for $N=8$.

The dependencies of $\tilde{x}_j\equiv x_j-Ct$ and $\dot{x}_j$ on $\tilde{\xi}_j\equiv(\xi_j-\beta/2)/(2\pi)$ for all possible oscillation modes are represented in Fig. 3 for $N=8$, $a=1$, and $\omega_0=1$. Oscillations of the velocities of balls are close in their shape either to a light autosoliton (for $n\leqslant 3$) or to a dark one (for $n\geqslant 5$). We see that the oscillations corresponding to the modes with $n$ close to unity and $n$ close to $N$ have large amplitudes and are essentially nonharmonic, whereas the oscillations corresponding to the modes with $n$ close to $N/2$ have moderately small amplitudes and are nearly harmonic. The qualitative explanation of these results can be given as follows. For $n$ close to unity and $n$ close to $N$, when the number of traveling solitons is small, each of them gets a moderately large amount of energy from the source. Owing to this fact each of the excited solitons has a large amplitude. Contrary, for $n$ close to $N/2$, when the number of traveling solitons is large, each of the solitons gets only a small amount of energy from the source and therefore has a small amplitude. For $a=1$ the oscillation frequency first increases, as $n$ increases from 1 to 4, and then decreases as $n$ increases from 4 to 7 [see Fig. 2(g)], whereas for $a=0.1$ the frequency has two minima for $n=3$ and $n=5$ [see Fig. 2(h)].

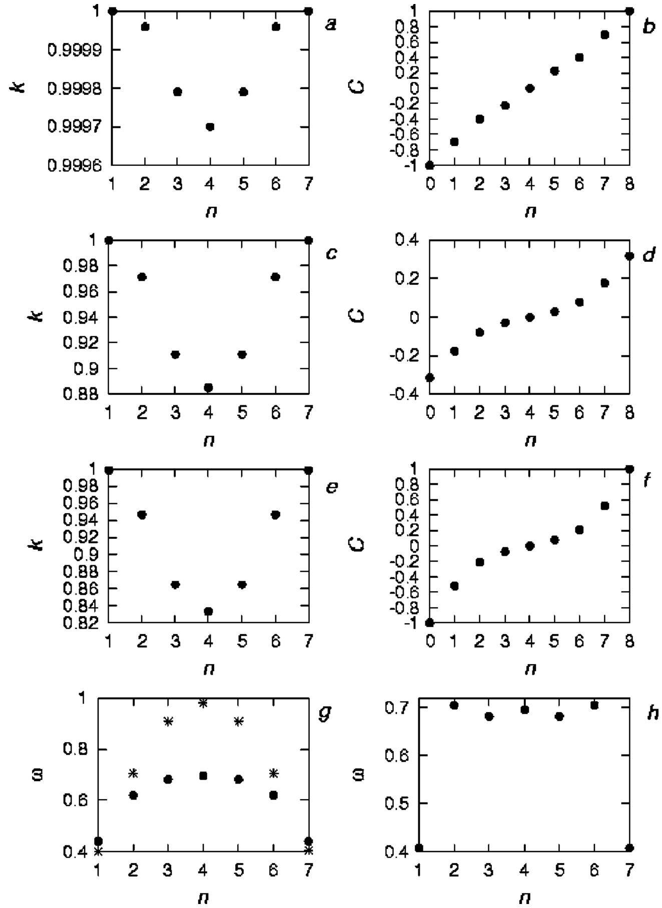


FIG. 2. The dependencies of the steady-state values of the modulus $k$ and of the constant constituent of velocity $C$ on $n$ for $N=8$: (a) and (b) $a=1,\omega_0=1$, (c) and (d) $a=0.1,\omega=1$, and (e) and (f) $a=1,\omega_0=4$; (g) and (h) the dependencies of the frequency $\omega$ on $n$ for (g) $a=1,\omega_0=1$ (circles) and $\omega_0=4$ (stars), and (h) $a=0.1,\omega_0=1$.

Numerical simulation of Eqs. (2) for $N=8$ shows that all of the modes indicated can be excited by means of the variation of initial conditions. As an example, we give a table of initial values of $\dot{x}_j$ such that different modes are excited, for $x_j(0)=0,\omega_0=1,\mu=0.1,a=1$ (see Table I).

Consider now Eqs. (4) and set dissipative forces in the form (6). For $\mu=0$ a partial solution of Eqs. (4) describing solitonlike oscillations is determined by Eq. (19). As in preceding case, a chain from $N$ elements described by Eqs. (4) possesses $N+1$ different modes of self-oscillations. These modes differ from one another by amplitudes, frequencies and phase shifts between the oscillations of strains of neighboring springs. It should be noted that two of these modes ($\dot{z}_j=\pm\sqrt{b}$) do not have a physical meaning because they

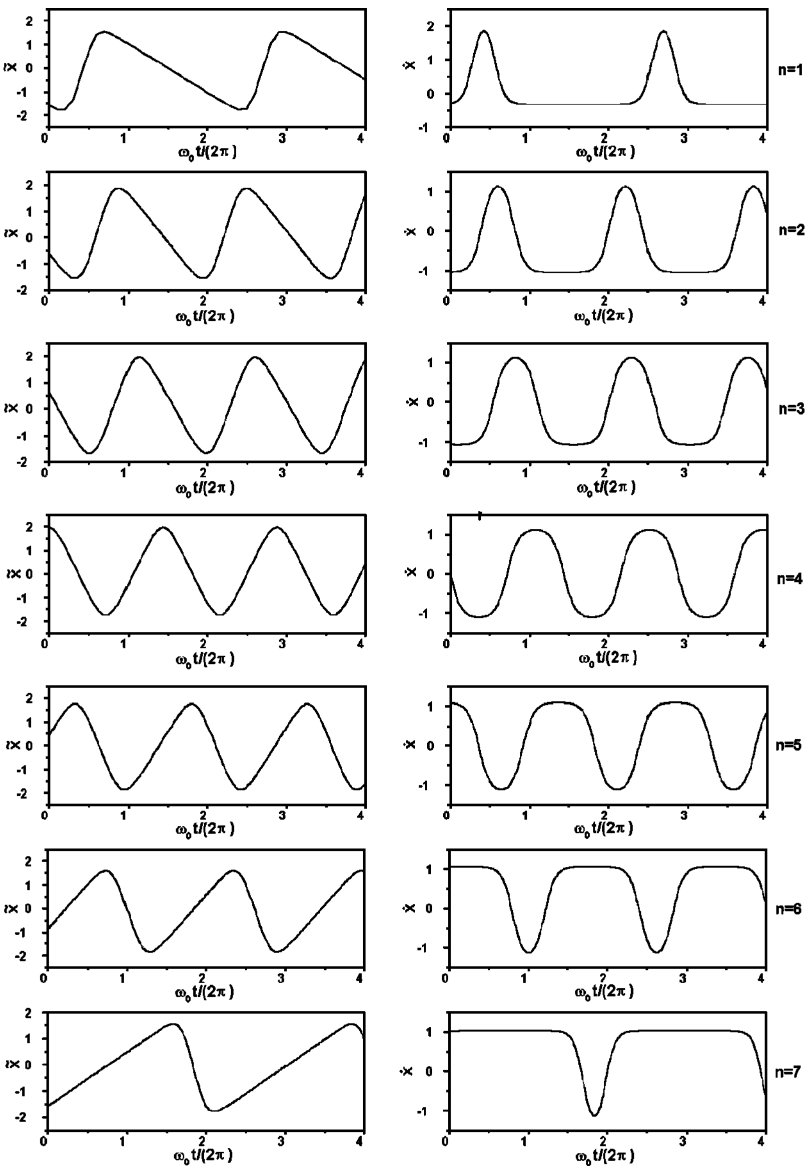


FIG. 3. The dependencies of $\tilde{x}_j \equiv x_j - Ct$ and $\dot{x}_j$ on $\tilde{\xi}_j \equiv (\xi_j - \beta/2)/(2\pi)$ for all possible modes of oscillations of one of the balls in the case of $N=8, a=1, \omega_0=1$.

correspond unbounded expansion or contraction of the chain. To calculate the modulus $k$ for different modes we can, as before, use the averaged energy conservation law. Multiplying, for each $j$, the $j$th equation from Eq. (4) by $\dot{z}_j$, adding all equations and averaging over time we obtain

$$\frac{d}{dt}\sum_{j=1}^{N}\int_0^{2\pi}\left(\frac{\dot{z}_j^2}{2}+\frac{1}{m}\{2f(z_j)+z_j[f(z_{j-1})+f(z_{j+1})]\}\right)d\xi_j = \mu\sum_{j=1}^{N}\int_0^{2\pi}(b-\dot{z}_j^2)\dot{z}_j^2 d\xi_j, \tag{32}$$

where

$$\dot{z}_j(\xi_j,k) = \frac{8\omega^3 k^2 \mathbf{K}^3(k)}{\pi^3\omega_0^2}\left\{1-\frac{4\omega^2\mathbf{K}^2(k)}{\pi^2\omega_0^2}\left[\frac{\mathbf{E}(k)}{\mathbf{K}(k)} - \mathrm{dn}^2\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right)\right]\right\}^{-1}\mathrm{sn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right) \times \mathrm{cn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right)\mathrm{dn}\left(\frac{\mathbf{K}(k)}{\pi}\xi_j,k\right), \tag{33}$$

TABLE I. An example of initial values of $\dot{x}_j$ such that different modes are excited.

| Mode number | $\dot{x}_1(0)$ | $\dot{x}_2(0)$ | $\dot{x}_3(0)$ | $\dot{x}_4(0)$ | $\dot{x}_5(0)$ | $\dot{x}_6(0)$ | $\dot{x}_7(0)$ | $\dot{x}_8(0)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |
| 1 | 1 | 1 | −1 | −1 | −1 | −1 | −1 | −1 |
| 2 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | −1 |
| 3 | 1 | −1 | 1 | −1 | 1 | −1 | −1 | −1 |
| 4 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| 5 | −1 | 1 | −1 | 1 | −1 | 1 | 1 | 1 |
| 6 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | 1 |
| 7 | −1 | −1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

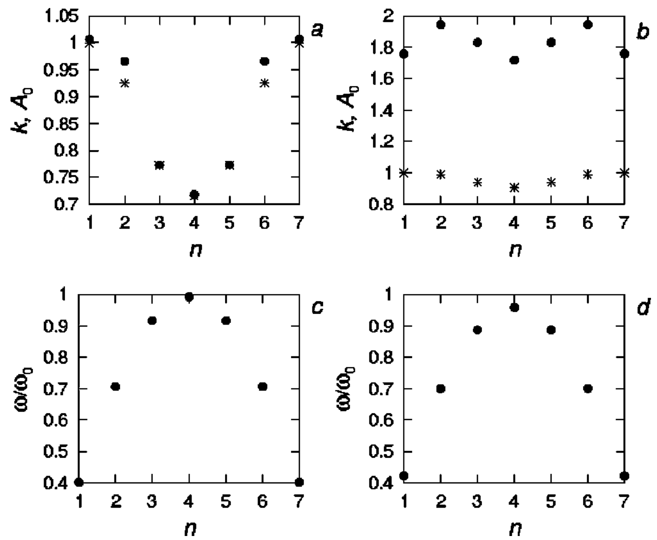


FIG. 4. The dependencies of the steady-state values of $k$ (stars), $A_0$ (circles) and $\omega/\omega_0$ on $n$ for $N=8$, (a),(c) $a=0.1$ and (b),(d) $a=0.5$.

In the steady-state regime the value of $k$ is determined from the equation

$$\int_0^{2\pi}(b-\dot{z}_j^2)\dot{z}_j^2 d\xi_j=0. \tag{34}$$

For every value of $k$ the integrals in Eq. (34) were calculated by us numerically and this equation was solved by means of graphical displays. The results of the calculations are illustrated in Fig. 4, where the dependencies of the steady state values of $k$ (stars), of the oscillation amplitudes $A_0=z_{\max}-z_{\min}$ (circles), and of the relative frequencies $\omega/\omega_0$ are shown.

The first four oscillation modes for all springs are represented in Figs. 5 and 6 for $N=8$. We see that the oscillations corresponding to the first (and the seventh) mode, which have the largest amplitude, are essentially nonharmonic; whereas the oscillations corresponding to the other modes are close to harmonic in their shape. The oscillation frequency first increases, as $n$ increases from 1 to 4, and then decreases as $n$ increases from 4 to 8.

Figure 7 illustrates the projections of limit cycles for different modes ($n=1,2,3,4$) on the planes $z_j$, $\dot{z}_j$ (a), and $z_j$, $z_{j+1}$ for $a=0.1$. It is seen that these projections depend essentially on the mode number.

## V. DISCUSSION OF APPLICATIONS AND GENERALIZATIONS OF THE DISSIPATIVE FORCES

The oscillatory modes analyzed here are not only theoretical constructions but they have been observed and studied already numerically and also experimentally by means an analog implementation consisting of $N\leqslant 6$ electrical circuits [17]. The implementation of Toda systems used in the cited work was based on diode double capacitor circuits which were proposed by Singer and Oppenheim [16]. In comparison to Singer and Oppenheim additional blocks were introduced modeling the dissipative energy sources. In fact it was shown in the cited experimental work [15–17] that the electrical implementation of exponential nonlinearities of Toda type is not a difficult task. Indeed the natural nonlinearity of diodes is of exponential type and active elements one finds in many electrical circuits. This, in fact gave us the motivation to go into deeper details of the theoretical analysis of active

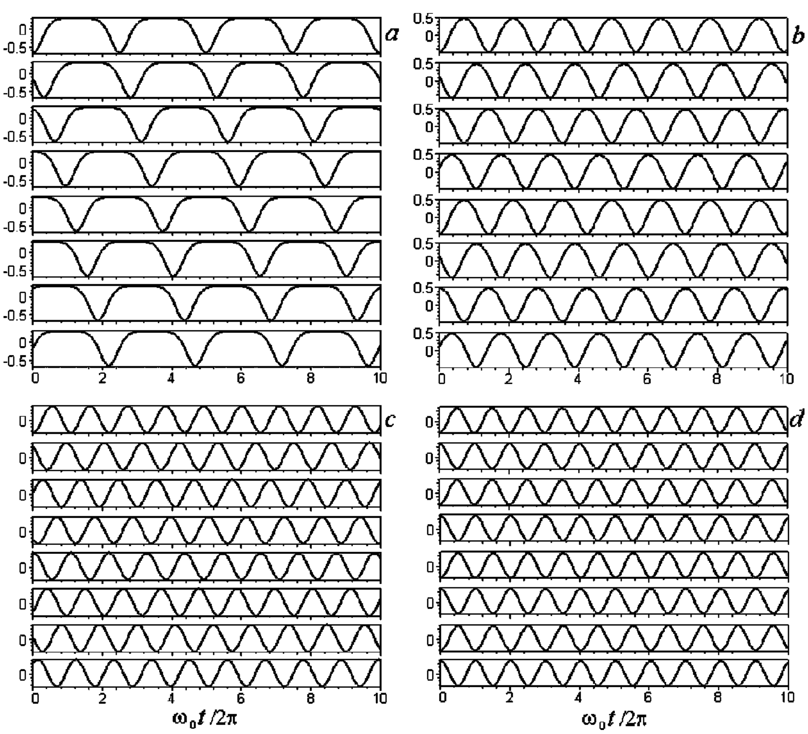


FIG. 5. The oscillations of all spring strains corresponding to the first four oscillation modes for $a=0.1$.

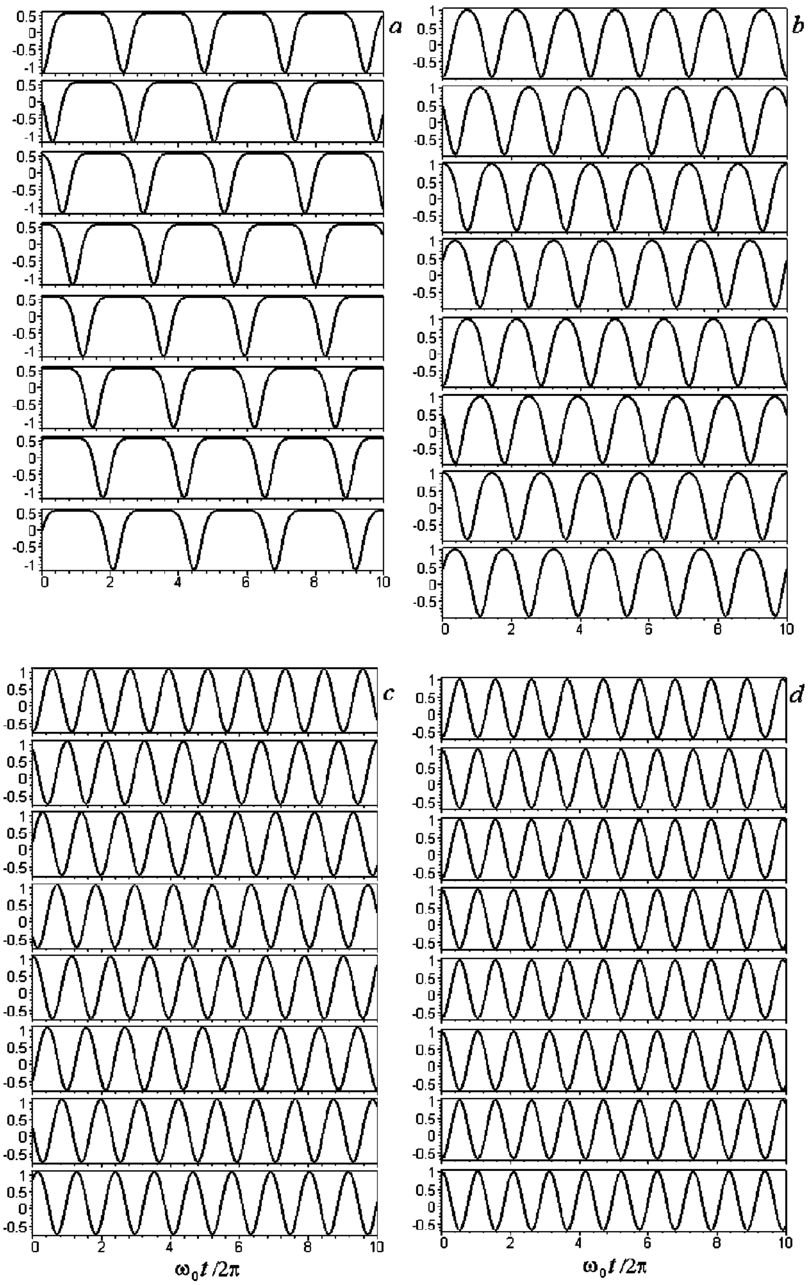


FIG. 6. The oscillations of all spring strains corresponding to the first four oscillation modes for $a=0.5$.

Toda systems and to look for explicit solutions. We believe that these systems might be even the prototypes for practical applications of nonlinear excitations.

We have shown above that exponential nonlinearities may quite easily be implemented by diodes, however, the implementation of the dissipative nonlinearities is not as easy and requires some effort [17]. Here we concentrated on dissipative nonlinearities of Rayleigh type. Let us discuss now a few possible generalizations of the dissipative forces, corresponding formally to different dissipative factors $g$. We will

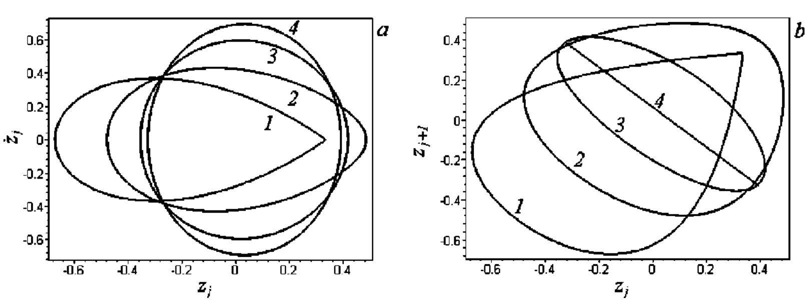


FIG. 7. The projections of limit cycles on the planes (a) $z_j, \dot{z}_j$ and (b) $z_j, z_{j+1}$ for the first (the curves 1), the second (the curves 2), the third (the curves 3), and the fourth modes (the curves 4).

discuss the following models.

Depot model of friction. Based on a physical model for the transfer of energy from an internal depot to acceleration of motion the following expression has been derived [12,14]:

$$g=\left(\gamma_0-\frac{\gamma_1}{1+\gamma_2\dot{x}_j^2}\right). \tag{35}$$

In the case of small oscillation amplitudes this expression coincides with the Rayleigh friction function considered in the preceding sections. It should be noted that a similar dissipative factor was derived by Monod [19] for the description of the growth of bacteria in nutrient medium.

Friction depending on the total momentum. Another interesting model is the following:

$$g=a-\overline{v}^2, \tag{36}$$

where

$$\overline{v}=\frac{1}{N}\sum_i v_i \tag{37}$$

is the average velocity of the balls.

Friction depending on the full energy. This model for dissipation factors has been studied in Ref. [10]. The assumption that the friction factor depends only on the full energy leads to

$$g=e-\frac{E}{N}, \tag{38}$$

where $E=H$ is the full energy determined by the Hamiltonian of the Toda chain. Such setting of the dissipation factor is similar to that in the Bautin equation [13].

The specific interest in the last two models of dissipation defined by Eqs. (36),(38) is connected with the property that these dissipation factors $g$ depend only on integrals of motion of the generative conservative system. This special property guarantees that any solution for the conservative case $\mu=0$ may be transferred to the dissipative case by adjusting the parameters $a$ and $e$ to the particular value of the corresponding integral. Then the dissipative terms have mainly the function to drive the system to the special value of the corresponding integral (which plays now the role of a characteristics of the attractor) and after certain relaxation time the system remains on the invariant set. It is evident that in the case of small dissipation the methods considered in the previous section are applicable for the calculation of the oscillation amplitudes and frequencies for all dissipative factors considered above.

## VI. CONCLUSION

We have shown that by using the averaged energy and momentum conservation laws the shape of self-oscillations and the values of the steady-state amplitudes and frequencies can be calculated. It is important that the generative soliton-like solutions, which can hardly be observed in the conservative chain (even in its numerical simulations) reveal themselves as small dissipation is present.

[1] M. Toda, Suppl. Prog. Theor. Phys. **45**, 174 (1970).
[2] M. Toda, Phys. Rep. **18**, 1 (1975).
[3] M. Toda, *Theory of Nonlinear Lattices* (Springer, Heidelberg, 1981).
[4] P.S. Landa and V.A. Vygodin, Kvant. Elektron. (Moscow) **4**, 769 (1977).
[5] M. Henon, Phys. Rev. B **9**, 1921 (1974).
[6] M. Henon and C. Heiles, Astrophys. J. **69**, 73 (1964).
[7] B.S. Kerner and V.V. Osipov, *Autosolitons: A New Approach to Problems of Self-Organization and Turbulence* (Kluwer, Dordrecht, 1994).
[8] C.I. Christov, M.G. Velarde, Physica D **86**, 323 (1995).
[9] H. Linde *et al.*, Phys. Fluids A **5**, 1068, 3162 (1993); J. Colloid Interface Sci. **188**, 16 (1997).
[10] V. Makarov, W. Ebeling, and M. Velarde, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, No. 5 (2000).
[11] W. Ebeling, U. Erdmann, J. Dunkel, and M. Jenssen, J. Stat. Phys. **101** (2000).
[12] F. Schweitzer, W. Ebeling, and B. Tilch, Phys. Rev. Lett. **80**, 5044 (1998).
[13] P.S. Landa, *Nonlinear Oscillations and Waves in Dynamical Systems* (Kluwer, Dordrecht, 1996).
[14] W. Ebeling, F. Schweitzer, and B. Tilch, BioSystems **49**, 17 (1999).
[15] P.S. Landa, *Self-Oscillations in Continuous Systems* (Nauka, Moscow, 1983) (in Russian).
[16] A.C. Singer and A.V. Oppenheim, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 571 (1999).
[17] V. Makarov, E. Del Rio, W. Ebeling, and M.G. Velarde, Phys. Rev. E. (to be published).
[18] E. Janke, F. Emde, and F. Lösch, *Tafeln Höherer Funktionen* (B.G. Teubner Verlagsgesellschaft, Stuttgart, 1960).
[19] J. Monod, Ann. Inst. Pasteur (Paris) **79**, 390 (1950).
[20] B.S. Kerner and V.V. Osipov, *Autosolitons* (Nauka, Moscow, 1991) (in Russian).
[21] X.L. Chu and M. Velarde, Phys. Rev. A **43**, 1094 (1991).